Pythagorean Triples

Theorem 1 (Pythagorean theorem) Given a right triangle $\triangle ABC$ where the lengths of the two shorter sides are a and b and the length of the hypotenuse is c,

$$a^2 + b^2 = c^2.$$

Related Content Standards - 8^{th} Grade

- Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions.
- Explain a proof of the Pythagorean Theorem and its converse.
- Apply the Pythagorean Theorem to find the distance between two points in a coordinate system.

Definition 1 A **Pythagorean triple** consists of three integers a, b, and c, such that

$$a^2 + b^2 = c^2$$

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Question 1 List all the Pythagorean triples you know.

Motivating Questions

- What interesting properties can we observe about Pythagorean triples?
- How can we construct a list of Pythagorean triples?
- How many Pythagorean triples are there?

Question 2 Prove if a, b, c is a Pythagorean triple, then ka, kb, kc is a Pythagorean triple for any $k \in \mathbb{N}$. What is the geometric interpretation of this Question?

Question 3 Find an isosceles right triangles with integer side lengths, or prove that no such triangle exists.

Question 4 Below is a list of Pythagorean triples.

- List any patterns you observe with the starred triples.
- Figure out what property all of the starred triples have in common.
- List any patterns you observe in the list of Pythagorean triples.

$3,4,5\star$	$27,36,45$	$24,70,74$
$6,8,10$	$14,48,50$	$21,72,75$
$5,12,13\star$	$30,40,50$	$45,60,75$
$9,12,15$	$24,45,51$	$30,72,78$
$8,15,17\star$	$20,48,52$	$48,64,80$
$12,16,20$	$28,45,53\star$	$18,80,82$
$7,24,25\star$	$33,44,55$	$13,84,85\star$
$15,20,25$	$40,42,58$	$36,77,85\star$
$10,24,26$	$36,48,60$	$40,75,85$
$20,21,29\star$	$11,60,61\star$	$51,68,85$
$18,24,30$	$16,63,65\star$	$60,63,87$
$16,30,34$	$25,60,65$	$39,80,89\star$
$21,28,35$	$33,56,65\star$	$54,72,90$
$12,35,37\star$	$39,52,65$	$57,76,95$
$15,36,39$	$32,60,68$	$65,72,97\star$
$24,32,40$	$42,56,70$	
$9,40,41\star$	$48,55,73\star$	

Conjectures:

Definition 2 A **primitive Pythagorean triple** consists of three integers a, b, and c such that

$$\gcd(a,b,c) = 1 \text{ and } a^2 + b^2 = c^2.$$

Question 5 The Pythagorean triple $(3,4,5)$ consists of consecutive integers. Are there any other Pythagorean triples that consist of consecutive integers? How many?

Question 6 What can you say about Pythagorean triples that form an arithmetic progression? That is what can you say about Pythagorean triples of the form $(x, x+k, x+2k)$?

Question 7 Take a few primitive Pythagorean triples (a,b,c) in the table compute the quantity $2c-2a$. Do these values seem to have some special form? Try to prove your observation is true for all primitive Pythagorean triples.

Question 8 Let m and n be numbers that differ by 2, and write the sum $\frac{1}{m}+\frac{1}{n}$ as a fraction in lowest term. For example, $\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$ and $\frac{1}{3}+\frac{1}{5}=\frac{8}{15}$.

- Compute the next three examples.
- Examine the numbers and denominators of the fractions and compare to the table of Pythagorean triples. Formulate a conjecture about such fractions.
- Prove that your conjecture is correct.

Question 9 Show that for any primitive Pythagorean triple exactly one of a,b, and c is odd.

Question 10 Show that for any primitive Pythagorean triple exactly one of a,b, and c is divisible by four.

Question 11 Show that for any primitive Pythagorean triple exactly one of a,b, and c is divisible by five.

Question 12

- Which odd numbers a can appear in a primitive Pythagorean triple (a,b,c)?
- Which even numbers b can appear in a primitive Pythagorean triple (a,b,c)?
- Which numbers c can appear in a primitive Pythagorean triple (a,b,c)?

Question 13 Show that the triple satisfies $a^2 = b + c$ if and only if $c - b = 1$.

Question 14 Show that every odd number appears in some primitive Pythagorean triple.

Question 15 Show that every multiple of four appears in some primitive Pythagorean triple.

Question 16

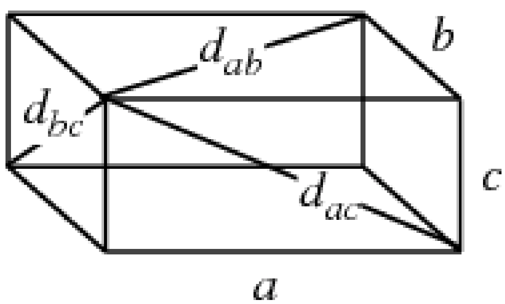
- How many Pythagorean triples are there with all three numbers prime?
- How many Pythagorean triples are there with at least two numbers prime?
- How many Pythagorean triples are there with at least one number prime?

Definition 3 An **Euler brick** is a cuboid that has integer edges $a > b > c$ and integer face diagonals:

$$d_{a,b} = \sqrt{a^2 + b^2}$$

$$d_{a,c} = \sqrt{a^2 + c^2}$$

$$d_{b,c} = \sqrt{b^2 + c^2}$$



An Euler brick is perfect if the surface diagonal is also an integer.

Question 17 Give an example of an Euler brick.

Question 18 Give an example of a perfect Euler brick.