Mathematical Empiricism and its Role in Education: a Case Study

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The Focus on Mathematics Academy

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is work in progress

THE FOCUS ON MATHEMATICS ACADEMY

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is work in progress
over the past 25 years . . .

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and still going strong!

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Core People Involved

Al Cuoco (Education Development Center)
Wayne Harvey (Education Development Center)
Steve Rosenberg (Boston University and MfA Boston)
Glenn Stevens (Boston University and MfA Boston)

and many others . . .

THE FOCUS ON MATHEMATICS ACADEMY

The Focus on Mathematics Academy is a Collaboration of

- Boston University, Boston College, UMass Lowell
- Education Development Center, Inc. (EDC)
- Math for America Boston (MfAB)
- Boston area school districts

working to support a community of

- Teachers,
- Educators, and
- Mathematicians

to close the gap between school mathematics and mathematics as a scientific discipline.

The Focus on Mathematics Academy

A community of mathematical practice that has been evolving for almost 25 years.

- A Targeted Math and Science Partnership (MSP)
  - established in 2003
  - with funding by the National Science Foundation
- Rooted in *PROMYS for Teachers*, initiated in 1989
- Phase II research beginning in 2009
- DRK12 funding to develop measures of secondary teachers' algebraic habits of mind
- Noyce Projects at Boston University and Boston College with MfA Boston in support of Master Teacher Leaders

WORK TO DATE: *Focus on Mathematics* PROGRAMS

The Focus on Mathematics Academy has established

- **Study Groups**
- **Seminars and colloquia**
- A graduate degree program at Boston University
  - Master of Mathematics for Teacing (MMT)
- Summer Institutes – e.g. **PROMYS**
- Mathematics fairs and Mathematics Expo for students
- Research collaboratives
- Tools for Assessing Secondary Teachers Algebraic Habits of Mind
- **Teachers in leadership roles throughout**

How do we define effective teaching?

"An effective mathematics curriculum is one that

- pays attention to students,
- is rich with mathematics, and
- finds a way of connecting the two."

–William J. McCallum, in testimony to the National Academy of Sciences

But how we implement this depends on our beliefs

- about students and how students learn,
- about mathematics, and
- about connections between the two.

Beliefs about students & student learning

- All students can achieve at high levels in mathematics;
- Students can enjoy doing mathematics;
- Effective teaching requires:
  - insight into how students think/reason/learn and solve problems;
  - understanding the “meaning” of student questions and developing strategies for mining student ideas.

Beliefs about the nature of mathematics

- Mathematics is natural
  - The empirical nature of mathematics
  - People do mathematics naturally
- Mathematics exists independent of us
  - We can perform experiments
  - We can test ideas and decide for ourselves
- Experience precedes formality
  - "Meaning" is determined by experience
  - Definitions and theorems are capstones
  - Language is a tool for coming to terms with experience
- Mathematics is the science of structure
  - Operations, order
  - Shape
  - Continuity
  - Transformation
- Mathematics is the art of figuring things out

The Common Core State Standards in Mathematics

Examining the whole mathematical enterprise

- as a coherent body of knowledge
- as a way of thinking and inquiring about the world we live in

Corresponding to these, there are two parts to the CCSSM:

- Content Standards
- Practice Standards

AUTHENTIC MATHEMATICAL EXPERIENCES

Experience first:

"It has been observed in every human activity experience comes first, and as this experience grows the need for communication motivates the development of language. Sadly enough, in our classroom practice we place language first and experience second. We worry about what we should say in order to help the student 'understand.' By this we mean to provide the effect of experience through the use of suitably chosen words. Not unexpectedly, the effect is at best a very pale image of the real thing."

Arnold Ross

Mathematical Habits of Mind

- **Acquiring experience**
  - numerical experimentation and alert observation
  - mathematics as an empirical science
  - practice – enhancing skills
  - inductive reasoning – building intuition and sense-making
- **Use of language**
  - precision
  - asking good questions, formulating conjectures
  - reasoning – proofs and disproofs
- **Review**
  - identifying important ideas
  - making sense of complex problems
  - looking for connections
- **Generalization**
  - broadening applicability
  - questioning answers

Some Odd Sums

Some Odd Sums

Some Odd Sums

$$1 = 1$$

Some Odd Sums

$$\begin{aligned} 1 &= 1 \\ 1+3 &= 4 \\ \cdots & \quad \cdots \end{aligned}$$

Some Odd Sums

$$\begin{aligned} 1 &= 1 \\ 1+3 &= 4 \\ 1+3+5 &= 9 \end{aligned}$$

Some Odd Sums

$$\begin{aligned} 1 &= 1 \\ 1+3 &= 4 \\ 1+3+5 &= 9 \\ 1+3+5+7 &= 16 \end{aligned}$$

Some Odd Sums

$$
\begin{aligned}
1 &= 1 \\
1+3 &= 4 \\
1+3+5 &= 9 \\
1+3+5+7 &= 16 \\
1+3+5+7+9 &= 25
\end{aligned}
$$

Some Odd Sums

$$
\begin{aligned}
1 &= 1 \\
1+3 &= 4 \\
1+3+5 &= 9 \\
1+3+5+7 &= 16 \\
1+3+5+7+9 &= 25 \\
1+3+5+7+9+11 &= 36
\end{aligned}
$$

Some Odd Sums

$$\begin{aligned}
1 &= 1 \\
1+3 &= 4 \\
1+3+5 &= 9 \\
1+3+5+7 &= 16 \\
1+3+5+7+9 &= 25 \\
1+3+5+7+9+11 &= 36 \\
1+3+5+7+9+11+13 &= 49 \\
\ldots &\quad \ldots
\end{aligned}$$

Some Odd Sums

$$
\begin{aligned}
1 &= 1 \\
1+3 &= 4 \\
1+3+5 &= 9 \\
1+3+5+7 &= 16 \\
1+3+5+7+9 &= 25 \\
1+3+5+7+9+11 &= 36 \\
1+3+5+7+9+11+13 &= 49 \\
\dots & \quad \dots \\
1+3+5+7+9+11+13+\cdots+(2n-1) &= ??
\end{aligned}
$$

Some Odd Sums

$$
\begin{aligned}
1 &= 1 \\
1+3 &= 4 \\
1+3+5 &= 9 \\
1+3+5+7 &= 16 \\
1+3+5+7+9 &= 25 \\
1+3+5+7+9+11 &= 36 \\
1+3+5+7+9+11+13 &= 49 \\
\ldots & \quad \ldots \\
1+3+5+7+9+11+13+\cdots+(2n-1) &= n^2
\end{aligned}
$$

Sums of Cubes

Sums of Cubes

Sums of Cubes

$$1^3 = 1$$

Sums of Cubes

$$
\begin{aligned}
1^3 &= 1 \\
1^3 + 2^3 &= 9
\end{aligned}
$$

Sums of Cubes

$$\begin{aligned} 1^3 &= 1 \\ 1^3 + 2^3 &= 9 \\ 1^3 + 2^3 + 3^3 &= 36 \end{aligned}$$

Sums of Cubes

$$\begin{aligned} 1^3 &= 1 \\ 1^3 + 2^3 &= 9 \\ 1^3 + 2^3 + 3^3 &= 36 \\ 1^3 + 2^3 + 3^3 + 4^3 &= 100 \end{aligned}$$

Sums of Cubes

$$
\begin{aligned}
1^3 &= 1 \\
1^3 + 2^3 &= 9 \\
1^3 + 2^3 + 3^3 &= 36 \\
1^3 + 2^3 + 3^3 + 4^3 &= 100 \\
1^3 + 2^3 + 3^3 + 4^3 + 5^3 &= 225
\end{aligned}
$$

Sums of Cubes

$$
\begin{aligned}
1^3 &= 1 \\
1^3 + 2^3 &= 9 \\
1^3 + 2^3 + 3^3 &= 36 \\
1^3 + 2^3 + 3^3 + 4^3 &= 100 \\
1^3 + 2^3 + 3^3 + 4^3 + 5^3 &= 225 \\
1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 &= ??
\end{aligned}
$$

Sums of Cubes

$$
\begin{aligned}
1^3 &= 1\\
1^3+2^3 &= 9\\
1^3+2^3+3^3 &= 36\\
1^3+2^3+3^3+4^3 &= 100\\
1^3+2^3+3^3+4^3+5^3 &= 225\\
1^3+2^3+3^3+4^3+5^3+6^3 &= 441\\
\ldots &\quad \ldots\\
1^3+2^3+3^3+4^3+5^3+6^3+\cdots+n^3 &= ??
\end{aligned}
$$

Sums of Cubes

$$
\begin{aligned}
1^3 &= 1^2 \\
1^3 + 2^3 &= (1+2)^2 \\
1^3 + 2^3 + 3^3 &= (1+2+3)^2 \\
1^3 + 2^3 + 3^3 + 4^3 &= (1+2+3+4)^2 \\
1^3 + 2^3 + 3^3 + 4^3 + 5^3 &= (1+2+3+4+5)^2 \\
1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 &= (1+2+3+4+5+6)^2 \\
\dots & \quad \dots \\
1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + \cdots + n^3 &= (1+2+3+\cdots+n)^2
\end{aligned}
$$

THE HALL OF 20,000 CEILING LIGHTS – CAR TALK

RAY: Imagine, if you will, that you have a long, long corridor that stretches out as far as the eye can see. In that corridor, attached to the ceiling are lights that are operated with a pull cord. There are gazillions of them, as far as the eye can see. Let's say there are 20,000 lights in a row. They're all off. Somebody comes along and pulls on each of the chains, turning on each one of the lights. Another person comes right behind, and pulls the chain on every second light.
TOM: Thereby turning off lights 2, 4, 6, 8 and so on.
RAY: Right. Now, a third person comes along and pulls the cord on every third light. That is, lights number 3, 6, 9, 12, 15, etcetera. Another person comes along and pulls the cord on lights number 4, 8, 12, 16 and so on. Of course, each person is turning on some lights and turning other lights off. If there are 20,000 lights, at some point someone is going to come skipping along and pull every 20,000th chain.

When that happens, some lights will be on, and some will be off. Can you predict which ones will be on? Think you know?

Decimal Expansions of Rational Numbers

Every rational number has a decimal expansion that either terminates or is eventually periodic. Find the decimal expansion of each of the following rational numbers:

- $\frac{1}{3}, \frac{2}{3};$
- $\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7};$
- $\frac{1}{13}, \frac{2}{13}, \frac{3}{13}, \ldots, \frac{12}{13};$
- $\frac{1}{89}, \frac{1}{9899}, \frac{1}{998999}, \ldots$

Pascal's Triangle

					1					
				1		1				
			1		2		1			
		1		3		3		1		
	1		4		6		4		1	
1		5		10		10		5		1

(1) What is the sum of the entries in the 7th row (i.e. the row at the bottom)? the 6th row? the 5th row? Any conjectures?

(2) Repeat the above experiment but this time take the alternating sum of each row. What do you see now?

(3) How many of the entries in the 7th row are odd? How many in the 6th row? the 5th row? Can you guess how many entries of the 100th row of Pascal's triangle are odd? How many entries in the 100th row do you think will be prime to 3? to 5? What's going on?

A Problem from George Polya

ABRACADABRA: In how many ways can one spell out "ABRACADABRA" by traversing the following diamond, always going from one letter to an adjacent one?

```
          A
        B   B
      R   R   R
    A   A   A   A
  C   C   C   C   C
A   A   A   A   A   A
  D   D   D   D   D
    A   A   A   A
      B   B   B
        R   R
          A
```